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The total chromatic number of split-indifference graphs[☆]

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ABSTRACT

The total chromatic number of a graph G, $\chi_T(G)$, is the least number of colours sufficient to colour the vertices and edges of a graph such that no incident or adjacent elements (vertices or edges) receive the same colour. The *Total Colouring Conjecture (TCC)* states that every simple graph G has $\chi_T(G) \leq \Delta(G)+2$, and it is a challenging open problem in Graph Theory. For both split graphs and indifference graphs, the TCC holds, and $\chi_T(G) = \Delta(G) + 1$ when $\Delta(G)$ is even. For a split-indifference graph G with odd $\Delta(G)$, we give conditions for its total chromatic number to be $\Delta(G) + 2$, and we build a $(\Delta(G) + 1)$-total colouring otherwise. Also, we pose a conjecture for a class of graphs that generalizes split-indifference graphs.

1. Introduction

Let G be a simple connected graph with vertex set $V(G)$ and edge set $E(G)$. An *element* of G is one of its vertices or edges. An edge $e \in E(G)$, whose ends are u and v, is denoted by $\{u, v\}$ or uv. An *edge colouring* is a map $\pi : E(G) \to \mathcal{C}$ with $\pi(e) \neq \pi(f)$ for any two adjacent edges $e, f \in E(G)$. If $\mathcal{C} = \{1, \ldots, k\}$, then π is a k-edge colouring. The smallest integer k for which a k-*edge colouring* exists is the *chromatic index* of G, denoted $\chi'(G)$. Clearly, $\chi'(G) \geq \Delta(G)$, where $\Delta(G)$ denotes the maximum degree of a vertex in G. Vizing's theorem [14] states that every simple graph G has an edge colouring with $\Delta(G) + 1$ colours. Therefore, either $\chi'(G) = \Delta(G)$ or $\chi'(G) = \Delta(G) + 1$. If a graph G has chromatic index $\Delta(G)$, then G is said to be *class* 1; otherwise, G is said to be *class* 2.

A *total colouring* is a map $\pi : E(G) \cup V(G) \to \mathcal{C}$ with $\pi(x) \neq \pi(y)$ for any two adjacent or incident elements $x, y \in E(G) \cup V(G)$. The smallest integer k for which a k-total colouring exists is the *total chromatic number* of G, denoted $\chi_T(G)$. Clearly, $\chi_T(G) \geq \Delta(G) + 1$. The *Total Colouring Conjecture (TCC)*, posed independently by Behzad [1] and Vizing [14], states that every simple graph G has a total colouring with $\Delta(G) + 2$ colours. By the TCC, either $\chi_T(G) = \Delta(G) + 1$ or $\chi_T(G) = \Delta(G) + 2$. If a graph G has $\chi_T(G) = \Delta(G) + 1$, then G is said to be *type* 1; if G has $\chi_T(G) = \Delta(G) + 2$, then G is said to be *type* 2. The TCC has been verified in restricted cases, such as graphs with maximum degree $\Delta(G) \leq 5$ [9], but the general problem has been open since 1964, exposing how challenging the problem of total colouring is.

The total colouring problem is known to be polynomial for few very restricted graph classes. The complexity of total colouring is open for the class of chordal graphs, and the partial results for the related classes of interval graphs [3], split graphs [5] and dually chordal graphs [7] expose the interest in the total colouring problem restricted to chordal graphs. It is NP-complete to determine whether the total chromatic number of an arbitrary graph G is $\Delta(G) + 1$ [13]. Note that this original NP-completeness proof was a reduction from the edge colouring problem, suggesting that, for most graph classes,

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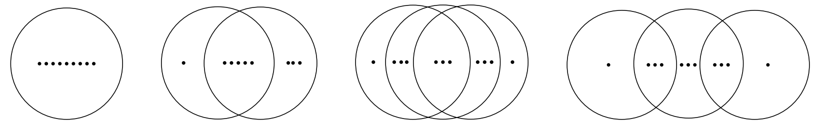

(a) Case (i). (b) Case (ii). (c) Case (iii). (d) Case (iii).

Fig. 1. The split-indifference graphs.

total colouring would be harder than edge colouring. The total colouring problem remains NP-complete when restricted to k-regular bipartite inputs [10], for each fixed $k \geq 3$. It is a natural step to investigate the complexity of total colouring restricted to classes for which the complexity of edge colouring is already established.

The motivation to investigate the total chromatic number of split-indifference graphs is twofold. On the one hand, it is the intersection of two graph classes for which the total colouring problem is still open. On the other hand, it is a graph class for which the edge colouring problem was solved [11].

A *clique* is a set of pairwise adjacent vertices, and an *independent set* is a set of pairwise non-adjacent vertices in the graph. A graph is a *split* graph if its vertex set can be partitioned into a clique and an independent set. A *unit interval* or *indifference* graph is the intersection graph of a set of unit intervals of a straight line. An *indifference* order of a graph is a total order on its vertex set such that the vertices of each maximal clique are consecutive with respect to the order. A graph is an indifference graph if and only if it admits an indifference order [12]. Split graphs and indifference graphs, with $\Delta(G)$ even, have $\chi_T(G) = \Delta(G) + 1$ [5,7]. Using a characterization of split-indifference graphs G, due to Ortiz et al. [11], the present work determines $\chi_T(G)$ when $\Delta(G)$ is odd. We give conditions which imply that $\chi_T(G) = \Delta(G) + 2$, and we build a $(\Delta(G) + 1)$-total colouring otherwise. We pose a conjecture for $\chi_T(G)$ when G has exactly three maximal cliques. This class of graphs can be viewed as a generalization of the split-indifference graphs.

2. Preliminaries

In 1967, Behzad et al. [2] proved that complete graphs with an even number of vertices are type 2 and those with an odd number of vertices are type 1. Let G be a graph with universal vertices. If G has an odd number of vertices, then it is type 1, since G is a spanning subgraph of a complete graph K_n, n odd. Otherwise, Theorem 1 establishes necessary and sufficient conditions for a graph G to be type 2.

Theorem 1 (*Hilton [8]*)**.** *Let G be a simple graph with an even number of vertices. If G has a universal vertex, then G is type* 2 *if and only if $\left|E(\overline{G})\right| + \alpha'(\overline{G}) < |V(G)|/2$, where $\alpha'(\overline{G})$ is the cardinality of a maximum independent set of edges of $\overline{G}$.*

Note that graphs with universal vertices and an even number of vertices satisfy the TCC because they are spanning subgraphs of a type 2 graph. Theorem 1 in fact characterizes the cases when graphs with even numbers of vertices and universal vertices are type 1 or type 2. Actually, Theorem 1 can be applied to a closed neighbourhood of a maximum degree vertex to determine cases for which a general graph G cannot be type 1. Therefore, we say that a general graph satisfies *Hilton's condition* if the subgraph induced by the closed neighbourhood of a maximum degree vertex is type 2.

Theorem 2 characterizes graphs that are simultaneously split and indifference in terms of maximal cliques.

Theorem 2 (*Ortiz Z. et al. [11]*)**.** *Let G be a connected graph, and let A, B, and C be sets of vertices. Graph G is split-indifference if and only if*

(i) *G is a complete graph; or*
(ii) *G is the union of two complete graphs $G[A]$ and $G[B]$, such that $G[A] \setminus G[B] \cong K_1$; or*
(iii) *G is the union of three complete graphs $G[A]$, $G[B]$, and $G[C]$, such that $G[A] \setminus G[B] \cong K_1$, $G[C] \setminus G[B] \cong K_1$, and $A \cup C = V(G)$ or $A \cap C = \emptyset$.*

Fig. 1 illustrates all possible cases of Theorem 2. A split-indifference graph without universal vertex must satisfy case (iii) when $A \cap C = \emptyset$. Therefore, by Theorem 1, to determine the total chromatic number of the whole class of split-indifference graphs, it remains to consider the case depicted in Fig. 1(d). The TCC has been established for split graphs [5] and for indifference graphs [7]. In fact, the total chromatic number is known for both classes when the maximum degree is even, but remains unknown for odd maximum degree. Therefore, for split-indifference graphs G, it only remains to establish $\chi_T(G)$ when G has no universal vertices and has odd maximum degree. The total chromatic numbers for split graphs and for indifference graphs with odd maximum degree are unknown.

3. Main results

In this section, graph G is a connected split-indifference graph without universal vertices and with odd maximum degree. Following the notation of Theorem 2, graph G has three maximal cliques, A, B, and C, such that $A \cap C = \emptyset$. Let $A' = A \setminus B$, $B' = B \setminus (A \cup C)$, $C' = C \setminus B$, $AB = A \cap B$, and $BC = B \cap C$. Note that $\left|A'\right| = \left|C'\right| = 1$. Therefore, the vertices of maximum degree of G are in the set $AB \cup BC$ and, with no loss of generality, we assume that $|AB| \geq |BC|$. We call (A', AB, B', BC, C') a *standard pseudo-partition* of $V(G)$.

Theorem 3. *Let G be a split-indifference graph without universal vertices and with odd maximum degree. Let (A', AB, B', BC, C') be a standard pseudo-partition. If $|AB| > |B'| + |BC| + 1$, then G is type 2.*

Proof. Recall that $\left|A'\right| = \left|C'\right| = 1$. Let $H = G[A \cup B]$. We conclude that $|V(H)|$ is even, because H has universal vertices, $\Delta(H) = \Delta(G)$, and $\Delta(G)$ is odd. By Theorem 1, graph H is type 2 if, and only if, $E(\overline{H}) + \alpha'(\overline{H}) < \frac{|V(H)|}{2}$. Note that $|V(H)| = 1 + |AB| + |B'| + |BC|$. Since all edges of $\overline{H}$ occur between the unique vertex of A' and the vertices of $B' \cup BC$, $|E(\overline{H})| = |B'| + |BC|$ and $\alpha'(\overline{H}) = 1$. Therefore, if $|AB| > |B'| + |BC| + 1$, then H is type 2. Recall that graph G satisfies the TCC. We conclude that

$$\Delta(G) + 2 = \Delta(H) + 2 = \chi_T(H) \leq \chi_T(G) \leq \Delta(G) + 2. \quad \square$$

Theorem 4. *Let G be a split-indifference graph without universal vertices and with odd maximum degree. Let (A', AB, B', BC, C') be a standard pseudo-partition. If $|AB| \leq |B'| + |BC| + 1$, then G is type 1.*

Proof. Assume that $|AB| < \left|B'\right| + |BC| + 1$. First, we define π, a colour assignment to the elements of $G[B]$, where $B = AB \cup B' \cup BC$. For each $v \in B$, $\pi(v) = i$, $1 \leq i \leq |AB| + \left|B'\right| + |BC|$, according to the following distribution: if $v \in AB$, then $1 \leq i \leq |AB|$; if $v \in BC$, then $|AB| + 1 \leq i \leq |AB| + |BC|$; if $v \in B'$, then $|AB| + |BC| + 1 \leq i \leq |AB| + |BC| + \left|B'\right|$.

Note that $\Delta(G) = |B| = |AB| + \left|B'\right| + |BC|$ and that each vertex of B receives a distinct colour in the set $\{1, \ldots, \Delta(G)\}$. Therefore, we can label the vertices of B with their colours. So as to complete the definition of π, we describe the colours of the edges of $G[B]$.

$$\pi(ij) = \begin{cases} \dfrac{i+j}{2}, & \text{if } (i+j) \equiv 0 (\text{mod } 2); \\ |B|, & \text{if } i+j = |B|; \\ \dfrac{i+j+|B|}{2}, \ \text{mod } |B| & \text{otherwise.} \end{cases}$$

Each edge of $G[B]$ receives a colour in the set $\{1, \ldots, \Delta(G)\}$. Clearly, π is a vertex colouring of $G[B]$. Also, $\pi(ij) \neq i$, for each edge ij with $i \neq j$; otherwise, $j \equiv i (\text{mod } |B|)$, which implies that $j = i$ or $j > |B|$. By the same reasoning, $\pi(ij) \neq \pi(ik)$ for each $i \neq j$, $i \neq k$, and $j \neq k$. Thus, π is a $\Delta(G)$-total colouring of $G[B]$.

Let $M := \{ij\colon 1 \leq i \leq |AB| \text{ and } j = i + |AB|\}$. Since $|AB| < \left|B'\right| + |BC| + 1$, $M \subseteq E(G[B])$ and M covers AB. Also, M covers BC, since the labels of vertices of BC are lower than the labels of vertices of B' and $|AB| \geq |BC|$. By the construction of π, the colours of the edges of $G[B]$ depend on the labels of their ends. By the construction of M, the sums of the ends of each edge of M form an increasing sequence from $|AB| + 2$ to $3\,|AB|$, with an increment of 2. By the definition of π, the colours obtained from this sequence are consecutive numbers. Moreover, $|M| < |B|$. Therefore, the colours of the edges of M are pairwise distinct.

Now, we assign colour $\Delta(G) + 1$ to vertices a and c. For the remaining edges we proceed as follows:

$$\pi(uv) = \pi(vw), \quad \text{where } u \in \{a, c\},\ v \in AB \cup BC, \text{ and } vw \in M; \tag{1}$$

$$\pi(vw) = \Delta(G) + 1, \quad vw \in M. \tag{2}$$

The colours assigned in (1) are different because they come from multi-coloured matching M. Moreover, in (2), the edges of matching M are recoloured with new colour $\Delta(G) + 1$.

In order to finish the proof, assume that $|AB| = \left|B'\right| + |BC| + 1$. Construct π, a $\Delta(G)$-total colouring for B, as we have done in the previous case, and let $M := \{ij\colon 1 \leq i \leq |AB| - 1 \text{ and } j = i + |AB|\}$. Note that, in this case, set M does not cover AB, because vertex $|AB|$ is not matched to any other vertex.

Set M is also a multi-coloured matching and, except for vertex $a \in A'$ and edge $\{a, |AB|\}$, π can be expanded to the elements of G, as was done previously. Edge $\{a, |AB|\}$ can receive colour $\Delta(G)+1$, since this colour was used only in elements of $M \cup \{c\}$. Vertex $a \in A'$ has $2\,|AB| = \Delta(G) + 1$ adjacent or incident elements. Thus, it is enough to show that two of them are coloured with the same colour.

Assume first that $|AB| \equiv 0 (\text{mod } 2)$. By construction, $\pi(\{a, 1\}) = |AB|/2 + 1$. Note that this colour was the colour of edge $\{1, |AB| + 1\} \in M$ before this edge received the colour $\Delta(G) + 1$. On the other hand, there exists one vertex in AB whose colour is $|AB|/2 + 1$. Thus, the number of colours that occur in a is less than $\Delta(G) + 1$.

The result for $|AB| \equiv 1 (\text{mod } 2)$ follows by similar arguments, considering edge $\{a, |AB| - 1\}$. $\square$

Corollary 5. *A split-indifference graph is type 2 if and only if it satisfies Hilton's condition.* $\square$

Table 1
Classes considered with respect to total colouring and their relation with Hilton's condition.

Graph class	Even Δ	Odd Δ
Complete	Type 1	Type 2 (Hilton's condition)
Universal vertex	Type 1	Hilton's condition
Split	Type 1	Open
Indifference	Type 1	Open
Split-indifference	Type 1	Hilton's condition (this work)
3-clique	Type 1	Open

4. The range of Hilton's condition

We state a conjecture about the total colouring of 3-clique graphs that can be seen as a generalization of the split-indifference graphs. A connected simple graph G is a *k-clique* graph if, and only if, G has exactly k distinct maximal cliques. If G is a 3-clique graph and has no universal vertex, then G is an indifference graph [6]. It remains to determine which 3-clique graphs G without universal vertices and with odd maximum degree are type 1 and which are type 2. This problem was first considered in [4], and some particular cases were solved. Moreover, in that work, the following conjecture was posed.

Conjecture 6. *A 3-clique graph is type* 2 *if and only if it satisfies Hilton's condition.*

For all graph classes listed in Table 1, every odd maximum degree graph is class 1 [5,6]. A more general question, which we leave open, is to determine the largest graph class for which all its odd maximum degree graphs are class 1 and for which all its even maximum degree graphs are type 1. A related question is to determine the largest graph class for which all its type 2 graphs satisfy Hilton's condition. A necessary condition for such a class is that its even maximum degree graphs are type 1.

References

[1] M. Behzad, Graphs and their chromatic numbers, Ph.D. Thesis, Michigan State University, 1965.
[2] M. Behzad, G. Chartrand, J.K. Cooper Jr., The colour numbers of complete graphs, J. London Math. Soc. 42 (1967) 225–228.
[3] V.A. Bojarshinov, Edge and total colouring of interval graphs, Discrete Appl. Math. 114 (2001) 23–28.
[4] C.N. Campos, O problema da coloração total em classes de grafos, Ph.D. Thesis, University of Campinas, 2006.
[5] B.L. Chen, H.-L. Fu, M.T. Ko, Total chromatic number and chromatic index of split graphs, J. Combin. Math. Combin. Comput. 17 (1995) 137–146.
[6] C.H. de Figueiredo, J. Meidanis, C.P. de Mello, On edge-colouring indifference graphs, Theoret. Comput. Sci. 181 (1997) 91–106.
[7] C.H. de Figueiredo, J. Meidanis, C.P. de Mello, Total-chromatic number and chromatic index of dually chordal graphs, Inform. Process. Lett. 70 (1999) 147–152.
[8] A.J.W. Hilton, A total-chromatic number analogue of Plantholt's theorem, Discrete Math. 79 (1990) 169–175.
[9] A.V. Kostochka, The total chromatic number of any multigraph with maximum degree five is at most seven, Discrete Math. 162 (1996) 161–163.
[10] C.J.H. McDiarmid, A. Sánchez-Arroyo, Total colouring regular bipartite graphs is NP-hard, Discrete Math. 124 (1994) 155–162.
[11] C. Ortiz, N. Maculan, J.L. Szwarcfiter, Characterizing and edge-colouring split-indifference graphs, Discrete Appl. Math. 82 (1998) 209–217.
[12] F.S. Roberts, On the compatibility between a graph and a simple order, J. Comb. Theory Ser. B 11 (1971) 28–38.
[13] A. Sánchez-Arroyo, Determining the total colouring number is NP-hard, Discrete Math. 78 (1989) 315–319.
[14] V.G. Vizing, On an estimate of the chromatic class of a *p*-graph, Diskret. Analiz. 3 (1964) 25–30 (in Russian).